A Brief Introduction to Computational Complexity

CS402 (Principles of Programming Languages)

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Overview

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Computational Complexity?

The study of the computational resources required for solving a given **problem**.

Examples of Problems:

- Given an integer, decide if it is prime.
- Given a set of integers, compute their average.
- Given a set of integers, sort them in ascending order.
- Given a map, find the shortest route from a to b.
- Given a set of constraints, find a solution that satisfies them all.
- Given a graph, decide if...
- ...it is connected.
- ...it admits a 3-coloring.
- ...it contains a clique of size 5.
- ...

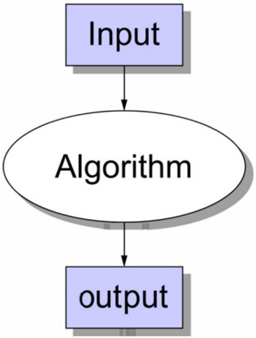
Definition of a Problem

Formally, a problem is a relation between two domains: the input domain and the output domain. Equivalently, it is a function from the input domain to the output domain.

Since computers (so far...) are discrete machines, both domains can be represented as sets of words over the $\{0, 1\}$ alphabet. Thus, a problem is a function from binary words to binary words:

$$\text{Problem } \Pi : \{0, 1\}^* \rightarrow \{0, 1\}^*$$

An algorithm solves Π if it computes $\Pi(x)$ for any $x \in \{0, 1\}^*$ (or in a prescribed subset of $\{0, 1\}^*$).



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Types of Problems

Different types of problems (examples based on coloring of a graph G):

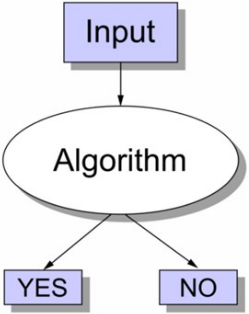
- **Decision:** Does G admit a k-coloring (for a given k)?
- **Search:** Find a k-coloring of G (for a given k).
- **Counting:** How many k-colorings does G admit (for a given k)?
- **Optimization:** What is the smallest k such that G admits a k-coloring?

Decision Problem (Focus)

A special case where $\Pi : \{0,1\}^* \rightarrow \{0,1\}$ (the answer is YES or NO).
$x \in \{0,1\}^*$ is a positive instance if $\Pi(x) = 1$ (resp. negative if 0).

Positive instances define a "formal language" (i.e., a set of words) $L \subseteq \{0,1\}^*$.

Point of View: Solving a decision problem boils down to deciding if a given word is in L.



Overview

What Questions?

Goal of computational complexity:

$\rightarrow$ classify the problems according to the resources needed for solving them.

What Type of Resources?

- **Time** (= number of operations)
- **Space** (= memory)
- **Randomness** (number of random bits)
- **Non-determinism**
- ...

Asymptotic Point of View

- What matters is how these quantities scale with the size of the input, noted n.
- Notations: $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, $o(\cdot)$, $\omega(\cdot)$
- Intuition: $\leq$, $\geq$, $=$, $<$, $>$ as $n \to \infty$, up to constant factors.
- Examples:
  - Constant: $O(1)$
  - Logarithmic: $O(\log n)$
  - Linear: $O(n)$
  - Quasi-linear: $O(n \log n)$
  - Quadratic: $O(n^2)$
  - Polynomial: $O(n^c) = n^{O(1)} = \text{poly}(n)$
  - Exponential: $O(2^n)$ or $O(2^{\text{poly}(n)})$
  - Factorial: $O(n^n)$

Overview

Space and Time

Unless otherwise mentioned, we focus on decision problems.

Generic Classes

- $\mathrm{TIME}(f(n))$: Set of problems solvable in time $O(f(n))$.
- $\mathrm{SPACE}(f(n))$: Set of problems solvable in space $O(f(n))$.

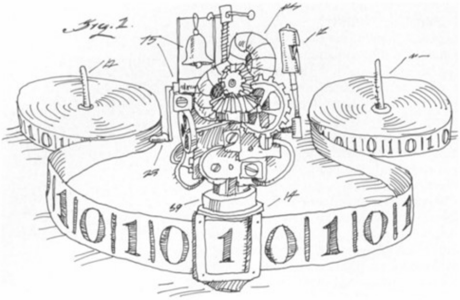
Main Classes of Complexity

$$\begin{aligned} L &:= \mathrm{SPACE}(\log n) && \text{Problems solvable in logarithmic space} \\ P &:= \mathrm{TIME}(\mathrm{poly}(n)) && \text{Problems solvable in polynomial time} \\ \mathrm{PSPACE} &:= \mathrm{SPACE}(\mathrm{poly}(n)) && \text{Problems solvable in polynomial space} \\ \mathrm{EXP} &:= \mathrm{TIME}(2^{\mathrm{poly}(n)}) && \text{Problems solvable in exponential time} \end{aligned}$$

$$L \subseteq P \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXP}$$

Machine Model?

- Turing machines
- Intuition: “standard algorithm” OK
- Universality?



Space versus Time (continued)

Theorems and Hierarchies (Simplified)

Given two functions $f_1(n)$ and $f_2(n)$ (abbreviated as f_1 and f_2 below):

- **Time hierarchy:**
  $f_1 = o(f_2 \log f_2) \Rightarrow \text{TIME}(f_1) \subsetneq \text{TIME}(f_2)$ (Stearns and Hartmanis '65)
- **Space hierarchy:**
  $f_1 = o(f_2) \Rightarrow \text{SPACE}(f_1) \subsetneq \text{SPACE}(f_2)$

Space versus Time?

$$\text{TIME}(f(n)) \subseteq \text{SPACE}\left(\frac{f(n)}{\log(f(n))}\right)$$ (Hopcroft, Paul, Valiant '77)

$\rightarrow$ *Space is strictly better than time!*

Overview

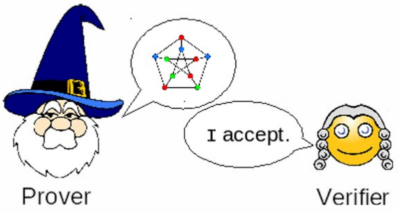
Non-determinism (NP and coNP)

Several equivalent definitions exist, with the simplest being:

- NP: $\exists$ short proof when the answer is YES (positive certificate)
- coNP: $\exists$ short proof when the answer is NO (negative certificate)

Short proof = one that is verifiable in polynomial time.

Example: 3-COLORATION $\in$ NP (certificate = the coloring itself)



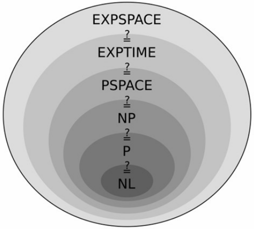
Non-determinism (continued)

More Generally

- NTIME($f(n)$): $\exists$ positive certificates verifiable in time $O(f(n))$.
- NSPACE($f(n)$): $\exists$ positive certificates verifiable in space $O(f(n))$.
- coNTIME and coNSPACE: Same for negative certificates.

Main Classes (Non-deterministic Version)

$$NL := \text{NSPACE}(\log n)$$
$$NP := \text{NTIME}(\text{poly}(n))$$
$$NPSPACE := \text{NSPACE}(\text{poly}(n))$$
$$NEXP := \text{NTIME}(2^{\text{poly}(n)})$$



Relations

- $P \subseteq NP \cap coNP$ (algorithm = verifier with an empty certificate)
- NSPACE($f(n)$) $\subseteq$ SPACE($f^2(n)$) (Savitch'70)
- NSPACE($f(n)$) = coNSPACE($f(n)$) (Immerman–Szelepcsényi'87)
- NTIME($f(n)$) $\subseteq$ SPACE($f(n)$) (Certificate enumeration)
- $L \stackrel{?}{=} NL \stackrel{?}{=} P \stackrel{?}{=} NP \stackrel{?}{=}$ PSPACE $\stackrel{?}{=}$ EXP $\stackrel{?}{=}$ NEXP $\stackrel{?}{=}$ EXPSPACE (Inclusions OK)
- Padding arguments: $P = NP \Rightarrow EXP = NEXP$ (and others)
- $P \stackrel{?}{=} NP$ Does "easy to check" imply "easy to compute"?

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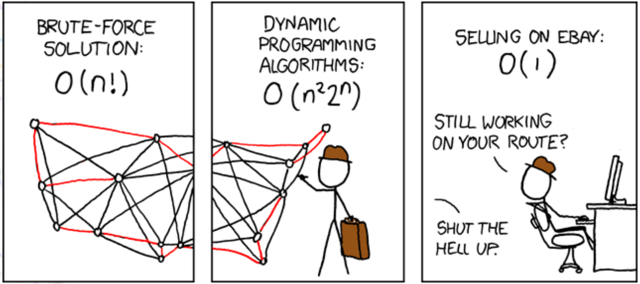
BRUTE-FORCE SOLUTION:
O(n!)
DYNAMIC PROGRAMMING ALGORITHMS:
O(n²2ⁿ)
SELLING ON EBAY:
O(1)
STILL WORKING ON YOUR ROUTE?
SHUT THE HELL UP.

Thanks!